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HOMOCLINIC BIFURCATION IN A SECOND ORDER DIFFERENTIAL EQUATION

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ABSTRACT

In this paper we have focused on homoclinic bifurcation in a second order nonlinear differential equation.

Bifurcation theory attempts to provide a systematic classification of the sudden changes in the qualitative behaviour of dynamical systems. A bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden qualitative change in its behaviour. Bifurcations are broadly classified into two types- local and global. Local bifurcation is associated with equilibria or cycles. Homoclinic bifurcation belongs to the global bifurcation category which deals with bifurcation events that involve larger scale behaviour in state space. A bifurcation which is characterized by the presence of trajectory connecting equilibrium with itself is called homoclinic bifurcation. Roughly speaking, a homoclinic orbit is an orbit of a mapping or differential equation which is both forward and backward asymptotic to a periodic orbit which satisfies a certain non-degeneracy condition called ‘‘hyperbolicity”.

The Melnikov method which uses Melnikov distance function provides a measure of the distance between a stable and unstable manifold. This method is used in our investigation.

KEYWORDS: Global Bifurcation, Stable and Unstable Manifold, Heteroclinic and Homoclinic Points and Orbits, Melnikov Method

1. INTRODUCTION

Bifurcation theory attempts to provide a systematic classification of the sudden changes in the qualitative behaviour of dynamical systems. It is the mathematical study of changes in the qualitative or topological structure of a given family. A bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden qualitative or topological change in its behaviour. The term originated with Poincare [26] in 1881. In order to understand the various types of qualitative behaviour that are exhibited by a physical system, it is necessary to describe the various bifurcations that occur in the system of differential equations modelling the physical system and to determine the parameter values, called bifurcation values, at which these bifurcations occur. Describing the bifurcations that occur in a system of differential equations is therefore an important, fundamental problem in the qualitative theory of differential equations.

Bifurcation theory is divided into two parts. The first part of the theory which is termed as local bifurcation theory [6, 10, 13, 29, 31], focuses attention on bifurcations that can be linked to the change in stability of either fixed points or limit cycles, which can be treated as fixed points of Poincare map [10, 29, 31]. In other words, local bifurcations are those in which fixed points or limit cycles appear, disappear, or change their stability. Since we can treat limit cycles as fixed

points of Poincare sections, we shall use the term fixed point bifurcation for both types. The change in stability is signalled by a change in the real part of one or more of the characteristic exponents of the jacobian matrix associated with that fixed point: In general the characteristic exponents can be complex numbers. At a local bifurcation, the real part becomes equal to 0 as some parameter (or parameters) of the system is changed. As the real part of the characteristic exponent changes from negative to positive, the motion associated with that characteristic direction goes from being stable (attracted toward the fixed point) to being unstable (being repelled by the fixed point). For a Poincare map fixed point, this criterion is equivalent to having the absolute value of the characteristic multiplier equal to unity. We call these bifurcations local because they can be analysed in terms of the local behaviour of the system near the relevant fixed point or limit cycle.

The other part of the theory, the part which is much less well-developed, deals with bifurcation events that involve larger scale behaviour in state space and hence are called global bifurcations. These global events involve larger scale structures such as basins of attraction [10, 14, 22, 32] and homoclinic orbits [1, 7, 16, 18, 21, 32] and heteroclinic orbits [1, 16, 18] for saddle points.

Global bifurcations are bifurcation events that involve changes in basins of attraction, homoclinic or heteroclinic orbits, or other structures that extend over significant regions of state space. Such bifurcations include intermittency and crises [13, 29] also. Since we need to take into account behaviour over a wide range of state space, a different means of classifying and studying such bifurcations is obviously needed. Global bifurcation cannot be detected through the analysis of the eigenvalues of the jacobian matrix associated with equilibria or cycles. Theory of global bifurcations is both more difficult and less articulated than in the theory of local bifurcations. This lack of development is due to the fact that in this case transition to chaos is not usually marked by any change in the fixed points of the system or the fixed points of a Poincare section. Specific cases, such as homoclinic tangencies and crises, have been studied in some detail [1, 9, 10, 12], but a general classification scheme is yet to be devised. A schematic classification of bifurcations involving chaotic attractors is given in [30, 31].

The rest of the paper is organized as follows. In section-2, we provide a review of stable, unstable manifolds, homoclinic orbit and homoclinic bifurcation. Section -3 deals with the analysis of the problem at our hand. In section-4 we have given a brief idea of Melnikov's method and have used it for detecting the parameter value at which a homoclinic orbit exist in case of the perturbed system we have considered. In section 5 we have shown that a homoclinic bifurcation occurs in the perturbed system. Section 6 includes our conclusions.

2. STABLE, UNSTABLE MANIFOLDS, HOMOCLINIC ORBIT AND HOMOCLINIC BIFURCATION

An invariant manifold is a surface contained in the phase space of a dynamical system which has the property that orbits starting on the surface stay on the surface throughout the course of their dynamical evolution. Knowledge of the invariant manifolds of a dynamical system as well as the intersections of their respective stable and unstable manifold is absolutely crucial in order to obtain a complete understanding of the global dynamics. The first rigorous results concerning invariant manifolds are due to Hadamard [11] and Perron [23, 24, and 25]. They proved the existence of stable and unstable manifolds of fixed points of maps and ordinary differential equations using different techniques. The existence of stable and unstable manifolds and their persistence under perturbation for an arbitrary manifold was first proved by Sacker [27]. This work was later extended and generalized by Fenichel [3, 4, and 5].

In the study of dynamical systems theory, the study of qualitative behaviour is emphasized; solutions of the

differential equation:

$$\dot{x} = f(x), \ \ x \in \mathbb{R}^n \tag{1}$$

Are viewed as flow-lines evolving in the state or phase space, $\mathbb{R}^n$. A key idea is that the behavior of a nonlinear system near a non-degenerate equilibrium or periodic orbit can be deduced by linearization and successive Taylor series approximations; geometrically, local stable and unstable manifolds exist. These manifolds are smooth (hyper-) surfaces, tangent at the equilibrium or periodic orbit to the eigenspaces belonging to exponentially decaying and growing linearized solutions, and invariant under the flow defined by equation (1). This is the main consequence of the stable manifold theorem. The local manifolds, which are related to nonlinear normal modes, are defined in a neighbourhood of the orbit in question, but they can be extended globally by following solutions backwards and forwards in time, and their structure determines the asymptotic behaviour of solutions starting nearby.

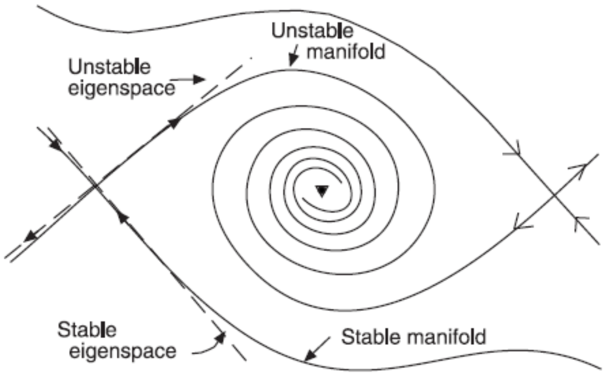

Figure 1: Stable and Unstable Manifolds for a Fixed Point: Equilibrium of a Free Nonlinear Oscillator, the Damped Pendulum [15]

Figure 1, shows the stable and unstable manifolds (separatrices, here) of the saddle point $(\theta, v) = (\pm\pi, 0)$ corresponding to the unstable equilibrium of the damped pendulum, whose governing equation can be written in non-dimensional form as

$$\dot{\theta} = v,$$

$$\dot{v} = -\sin\theta - \delta v$$

Note that the local stable manifold of the downward equilibrium $(\theta, v) = (0, 0)$ includes a full neighbourhood of that point: it has no unstable manifold; indeed, almost all solutions eventually approach$(0, 0)$; those that do, belong to its domain of attraction. In the above, `non-degenerate' means that all eigenvalues of the system linearized at the fixed point have nonzero real parts; such points are also called hyperbolic. Both equilibria are hyperbolic in Figure 1.

It is to be noted that if the governing equation of the pendulum can be written in non-dimensional form as

$$\dot{\theta} = v,$$

$$\dot{v} = -\sin\theta$$

i.e. (neglecting the term $(-\delta v)$ in the second equation), then the phase portrait will be as follows:

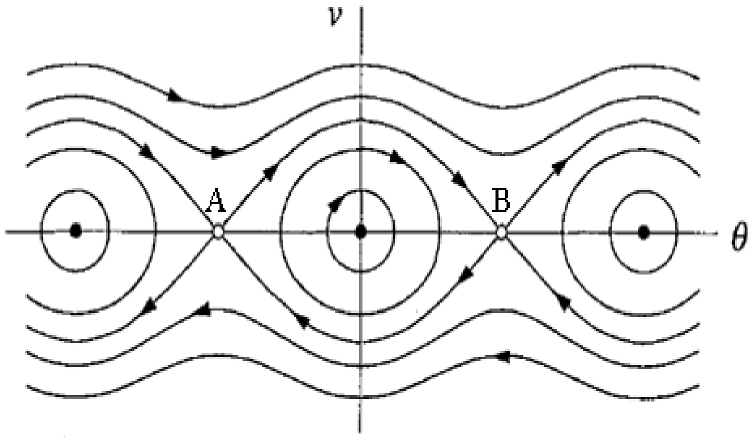

Figure 2: The Resulting Phase Portrait of the above System [29]

In this case, the unstable manifold of the saddle point at 'A' merge with the stable manifold of the other saddle point at 'B'. In this case, the points A and B are termed as heteroclinic points and they are said to be connected by a heteroclinic orbit. It may happen that the points A and B are same in certain cases i.e. for certain value of the system parameter it may be possible to find initial points on the unstable manifold that are such that the orbit hits the stable manifold. Such an orbit will, after an infinite amount of time, hit the fixed point from whose neighbourhood it once started out. This kind of orbit is called homoclinic orbit. They are common in conservative systems, but are rare otherwise [15, 29]. The qualitative behaviour of the system is found to be entirely different at parameter values before and after where homoclinic orbit is formed. For this reason it is termed as a homoclinic bifurcation.

Homoclinic orbits were first defined by Poincare in his treatise on the ''restricted three-body problem'' [26]. Further advances were made by Birkhoff [2] and by Smale [28]. Since that time, they have been studied by many researchers and have been shown to be intimately related to our understanding of nonlinear dynamical systems. There are many systems which possess homoclinic orbits. In one striking example discussed in the book of Moser [20], it was shown how homoclinic orbits can be used to account for the unbounded oscillatory motion discovered by Shilnikov in the three-body problem.

3. STUDY OF OUR MODEL

We are interested in the study of the system given by

$$\ddot{x} - x + x^3 - \varepsilon(\lambda y + x^2 y) = 0 \tag{2}$$

Where λ is the system parameter and ε is a perturbation parameter

It is already known that the system $\ddot{x} - x + x^3 = 0$ (i.e. if we make the perturbation parameter $\varepsilon = 0$) has a Homoclinic orbit [8, 10, 13].

Our aim is to investigate whether this type of orbit still exist if we perturb the system so that it becomes as given in (2) and if so for what value of the parameter λ.

Before proceeding further, let us see in detail about the existence of the homoclinic orbit in the unperturbed system$\ddot{x} - x + x^3 = 0$. It is important to note at this point that the system is a conservative one where the potential is given by $V = \frac{x^4}{4} - \frac{x^2}{2} + C$.

For further investigation, we convert the equation of the unperturbed system to two first order equations which are given by

$$\dot{x} = y \qquad \dot{y} = x - x^3 \tag{3}$$

The equilibrium points are found by putting $(\dot{x}, \dot{y}) = (0,0)$ which are found to be $(0,0)$ and $(\pm 1,0)$.

Now, to study the behaviour of the system locally near the equilibrium points, we find the jacobian matrix which is found to be

$$A = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix}$$

So, the jacobian matrix for the equilibrium point which is at the origin$(0,0)$, is found to be

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

As the eigenvalues of the above jacobian matrix are found to be $\lambda_1 = 1$ and $\lambda_2 = -1$ i.e. both real with opposite signs, we conclude that the origin is a saddle point. The manifolds are determined by the eigenvectors corresponding to these eigenvalues. The eigenvector for λ_1 and λ_2 are found to be $\{-1, 1\}^T$ and $\{1, 1\}^T$ respectively.

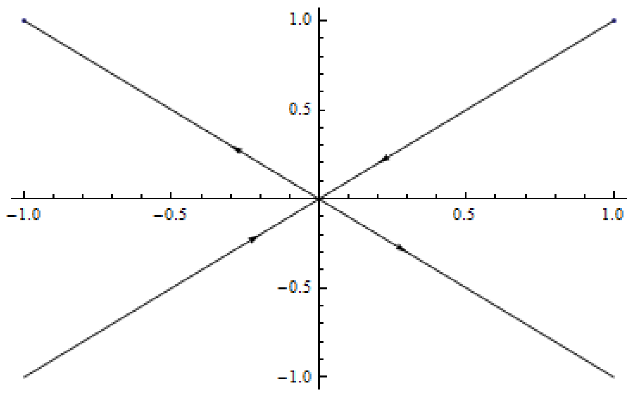

Figure 3: The Stable and Unstable Eigenvectors for the Unperturbed System at the Origin

The jacobian matrix at the other two equilibrium points$(x^*, y^*) = (\pm 1, 0)$, are found to be same and is given by

$$B = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$$

The eigenvalues of B are found to be $\lambda_1 = i\sqrt{2}$ and $\lambda_2 = -i\sqrt{2}$ i.e. purely imaginary. So, we conclude that we have centres at those equilibrium points. The manifolds are determined by the eigenvectors corresponding to these eigenvalues because the manifolds will be tangential to these eigenvectors near the equilibria. The eigenvector for λ_1 and λ_2 are respectively found to be $\{-\frac{i}{\sqrt{2}}, 1\}^T$ and$\{\frac{i}{\sqrt{2}}, 1\}^T$. At this point it is important to mention that drawing conclusion from purely imaginary eigenvalues is quite risky because even slightest deviation from this case can lead us to situations which are qualitatively completely different. So, it requires further computational verification. But in our case it is quite safe in drawing the same conclusion as the unperturbed system is a conservative one.

Below, in figure 5, we have shown the phase portrait of our considered unperturbed system which is drawn using the Runge-Kutta 4th order method. The direction of motion on the trajectories can be determined by drawing a direction field of the system (3) as shown in figure 4 below or by evaluating $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at one or two selected points. In fig.5 the critical point $(0,0)$ is a saddle point and the point $(-1,0)$and $(1,0)$ is centres which we already detected with the help of the nature of eigenvalues of the jacobian matrix at respective points.

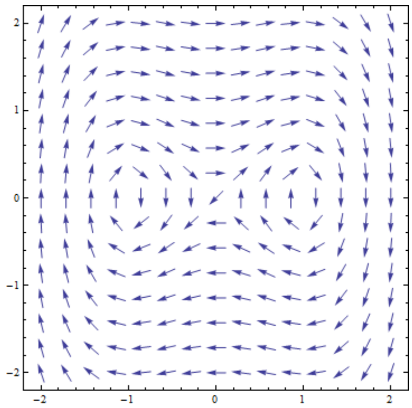

Figure 4: Vector Plot for $\ddot{x} - x + x^3 = 0$.

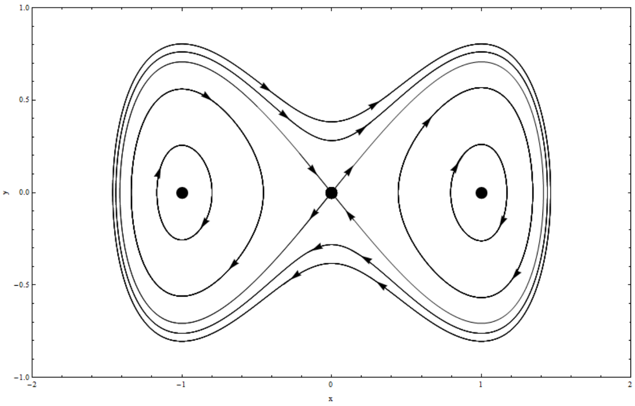

Figure 5: Phase Portrait for $\ddot{x} - x + x^3 = 0$

It is further noteworthy that the same phase portrait can be generated from the consideration of level surfaces as our system is a conservative one.

From the phase portrait, it is seen that each of the neutrally stable centres at $(\pm 1, 0)$ is surrounded by family of small closed orbits. At this point it is important to mention that physically, the unperturbed system represents motion of a particle in a double-well and these closed loops indicate periodic oscillatory motions in each of the wells individually. There are also large closed orbits that encircle all three fixed points. They correspond to periodic oscillatory motions which involve both the wells. Thus, solutions of the system are typically periodic, except for the equilibrium solutions and two very special trajectories: these are the trajectories that appear to start and end at the origin. More precisely, these trajectories approach the origin as $t \to \pm\infty$ which can mathematically be verified as follows:

The phase paths satisfy the separable differential equation

$$\frac{dy}{dx} = \frac{x - x^3}{y} \tag{4}$$

Solution of which gives

$$y^2 = x^2 - \frac{x^4}{2} + C, \tag{5}$$

Where C is an arbitrary constant?

Now the time solutions for these special trajectories which appear to start and end at the origin can be found by integrating

$$\left(\frac{dx}{dt}\right)^2 = x^2 - \frac{1}{2}x^4 \quad \text{(Putting} C = 0)$$

$$\Rightarrow \frac{dx}{dt} = x\sqrt{1 - \frac{1}{2}x^2}$$

$$\Longrightarrow \int \frac{dx}{x\sqrt{1-\frac{1}{2}x^2}} = \int dt \quad (6)$$

Substituting $x = \pm\sqrt{2}\, sechu$ and $dx = \mp\sqrt{2}\, sechu\ tanhu\ du$ in (6), we have

$$\int \frac{\sqrt{2}\, sechu\ tanhu}{\sqrt{2}\, sechu\ \sqrt{1-sech^2u}} du = \int dt$$

$$\Longrightarrow \int \frac{tanhu}{tanhu} du = \pm(t - t_0)$$

$$\Longrightarrow u = \pm(t - t_0)$$

Hence solutions are

$x = \pm\sqrt{2}\, sech(t - t_0)$ For any t_0

So, when $t \to \pm\infty$ we have$x \to 0$.

These two very special trajectories are the homoclinic orbits. Apparently, though these orbits also look like closed trajectories, it is to be noted that a homoclinic orbit does not correspond to a periodic solution, because the trajectory takes forever trying to reach the point. It is observed that one trajectory leaves the saddle point (at $t \to -\infty$), loops around the center and returns to the saddle point (at $t \to \infty$).

Our next concern is for what value of the parameter these homoclinic orbits appear in the perturbed system given by (1). For finding out the solution to this question we adopt the Melnikov method[1, 19, 22, 32], a perturbative method for detecting homoclinic orbits in nonlinear systems that have smooth separatrices connecting saddle points prior to perturbation.

4. MELNIKOV'S METHOD FOR DETECTING HOMOCLINIC BIFURCATION AND ITS APPLICATION TO OUR PROBLEM

Melnikov developed a perturbative method for detecting homoclinic orbits in nonlinear systems that have smooth separatrices connecting saddle points prior to perturbation.

This is a global perturbation method applicable to systems which have a known homoclinic path in an underlying autonomous system. The system is then perturbed and conditions for which stable and unstable manifolds intersect are determined to leading order. There are various versions of the theory of increasing generality but here we have followed the method given in [17] where systems of the form

$$\dot{x} = y, \quad \dot{y} + f(x) = \varepsilon h(x, y, t) \quad (7)$$

Where $h(x, y, t)$ is T −periodic in t and $|\varepsilon|$ is a small parameter is considered.

The unperturbed system is

$$\dot{x} = y, \qquad \dot{y} + f(x) = 0$$

The basic idea involved with the method can be understood with the help of the following figure.

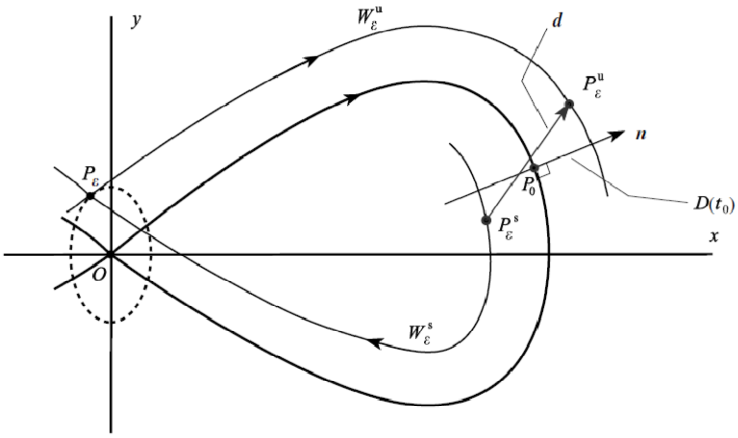

Figure 6: Distance Function $D(t_0)$ between the Unstable Manifold W_ε^u and The Stable Manifold W_ε^s at the Point P_0 [17]

The homoclinic orbit which exists in the unperturbed system ceases to exist due to perturbation as the stable and unstable manifolds get separated. Melnikov found the distance between these two manifolds which was named as Melnikov distance function and it is given by

$$D(t_0) = \int_{-\infty}^{\infty} y_0(t-t_0)\, h[x_0(t-t_0), y_0(t-t_0), t]dt$$

Clearly, when Melnikov distance function is found to be zero, then the stable and unstable manifolds get merged and we get a homoclinic orbit. This technique is used in case of our considered perturbed model. For finding the distance function for the problem at our hand, we consider the perturbed planar system of the form (7) which was given by

$$\dot{x} = y, \qquad \dot{y} + (-x + x^3) = \varepsilon(\lambda y + x^2 y) \tag{12}$$

Comparing the system (12) with (7), we get

$$f(x) = -x + x^3, \quad h(x,y,t) = \lambda y + x^2 y$$

Hence the Melnikov function,

$$D(t_0) = \int_{-\infty}^{\infty} y_0(t-t_0) h[x_0(t-t_0), y_0(t-t_0), t]dt$$

$$= \int_{-\infty}^{\infty}\{-\sqrt{2}\operatorname{sech}(t-t_0)\tanh(t-t_0)\}[\lambda\{-\sqrt{2}\operatorname{sech}(t-t_0)\tanh(t-t_0)\} + \{\pm\sqrt{2}\operatorname{sech}(t-t_0)\}^2\{-\sqrt{2}\operatorname{sech}(t-t_0)\tanh(t-t_0)\}$$

$$= 2\lambda \int_{-\infty}^{\infty} \operatorname{sech}^2(t-t_0)\, tanh^2(t-t_0)\, dt + 4\int_{-\infty}^{\infty} \operatorname{sech}^4(t-t_0)\, tanh^2(t-t_0)\, dt$$

Substituting $t - t_0 = u$, we get

$$D(t_0) = 2\lambda \int_{-\infty}^{\infty} \operatorname{sech}^2 u\, tanh^2 u\, du + 4\int_{-\infty}^{\infty} \operatorname{sech}^4 u\, tanh^2 u\, du$$

$$= 2\lambda\left(\frac{2}{3}\right) + 4\left(\frac{4}{15}\right) = \frac{4}{3}\left(\lambda + \frac{4}{5}\right)$$

For the existence of homoclinic orbits, the Melnikov distance function should be zero and hence for our perturbed system the homoclinic trajectories exist for the parameter value$\lambda = -\frac{4}{5} = -0.8$.

5. DETECTION OF HOMOCLINIC BIFURCATION IN OUR MODEL

Next, we investigate the behaviour of the system for parameter values smaller and greater than this particular parameter value where the homoclinic orbits appear. This was done by drawing phase portraits of the system for different parameter values which are shown along with the respective figures.

The phase portraits clearly show that if the parameter values are less than the parameter value at which the homoclinic orbits appear, the trajectories of the system spiral inward showing a trend to converge to a fixed point whereas the trajectories diverges to infinity if the parameter values are bigger than the value at which homoclinic orbits appear. Moreover, the homoclinic orbits ceases to exist. Thus, different qualitative behaviours are seen in the system for parameter values which are greater and smaller than the parameter value at which homoclinic orbits appear. Hence, we conclude that a homoclinic bifurcation takes place in the perturbed system at the parameter value$\lambda = -0.8$.

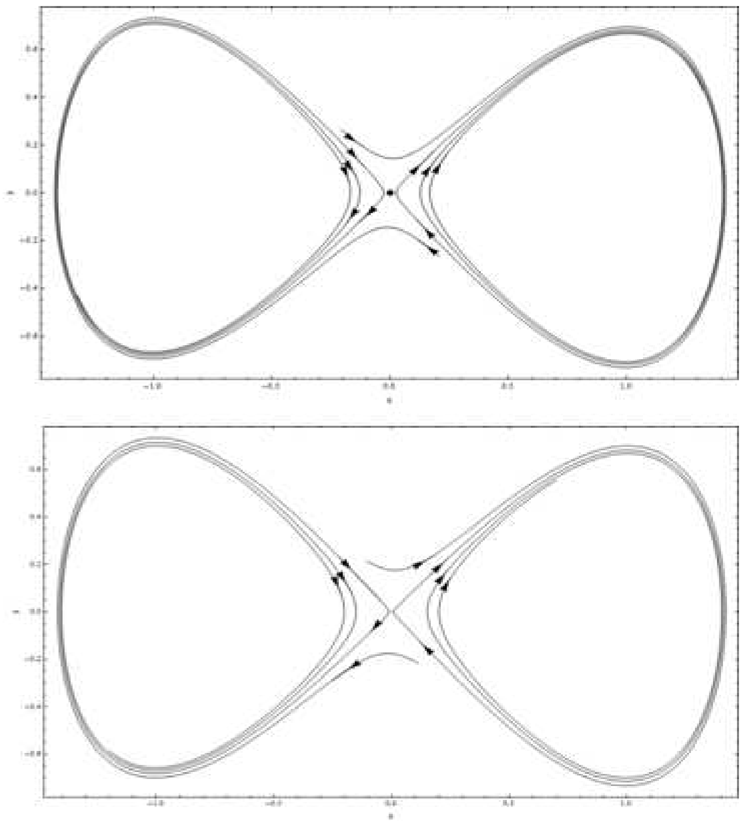

Figure 7: Phase Portraits for Parameter Value (-0.9) and (-0.87).

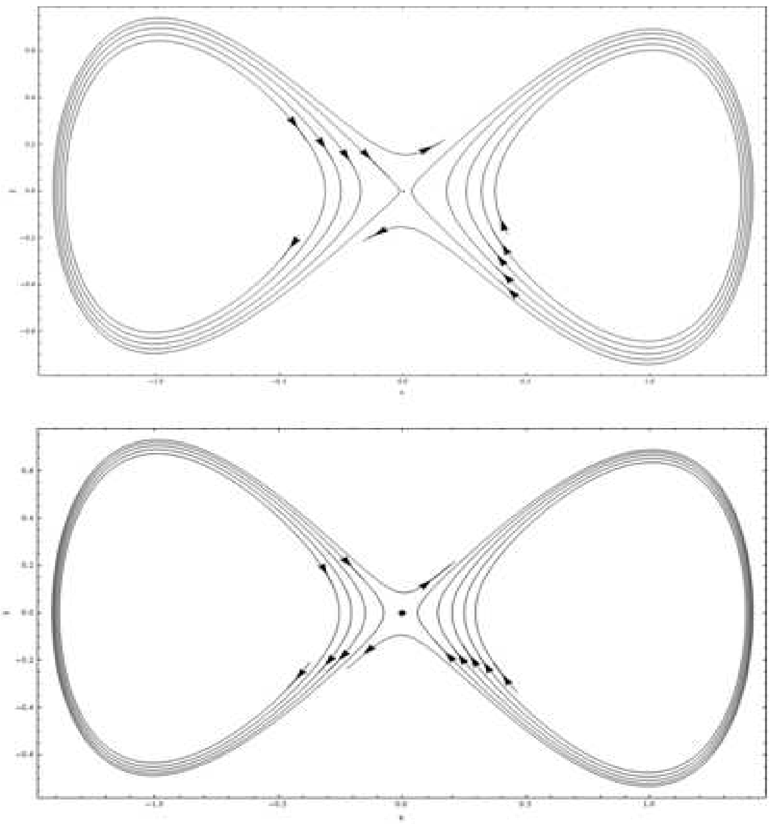

Figure 8: Phase Portraits for Parameter Value (-0.7) and (-0.75)

6. CONCLUSIONS

We found out the parameter value at which homoclinic orbits appear for the perturbed system with the help of Melnikov distance function method after knowing that homoclinic orbits exist in the unperturbed system. With the help of phase portrait we have shown that a homoclinic bifurcation takes place in the system at the parameter value at which homoclinic orbits appear in the system.

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